Moving dipoles and the relativity of simultaneity

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An observer moving parallel to a current-carrying wire detects an electric field due to the Lorentz transformation directed either toward or away from the wire, depending on the relative motion of observer and current. The accepted interpretation of this situation as viewed from the observer's rest frame is that there is a net linear charge density on the wire: The Lorentz contraction of the separation of fixed ions and charge carriers is different due to their different speeds in the observer's frame. The idea that a net charge exists on a wire in a reference frame moving parallel to the wire leads to the expectation that there is a charge separation seen on a moving current loop, resulting in paradoxes such as that proposed by Mansuripur. I argue that the apparent charge on a current-carrying wire is due to the relativity of simultaneity. The phenomenon of Lorentz contraction is a consequence of this effect such that there is no charge separation on a moving current loop. Given this insight, the nature of the fields of moving dipoles and the nature of the magnetization-polarization tensor are investigated.

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I. INTRODUCTION

The presence of a radial electric field centered on a current-carrying wire detected by observers moving parallel to the wire has been thought to be due to a net linear charge density in the wire detected by the observers. This charge density is supposed to be a result of the difference in the separation between positive charges versus negative charges in the wire due to Lorentz contraction, since these charges are observed to be traveling at different speeds [1, 2]. In the reference frame at rest with the wire, the charge carriers are in motion and the fixed ions are stationary, but the wire is taken to be neutral. In a frame moving parallel to the wire both types of charges are in motion, but one is moving faster than the other. The idea is that an observer moving parallel to the wire (or with a component of velocity parallel to the wire) will see a difference in the Lorentz contraction of the average separation of the two types of charges, with the faster charges appearing closer together than the slower ones.

This effect is what I call "differential Lorentz contraction". According to this idea, the difference in average separation results in a net linear charge density on the wire in the moving reference frame, which produces the radial electric field felt by a charge moving parallel to the wire. Hence, a readily observable phenomenon – the force of attraction or repulsion acting between parallel wires carrying currents – is thought to be a direct manifestation of length contraction which would otherwise might not be seen except in objects traveling near the speed of light.

The Lorentz transformation of the wire's magnetic field from its rest frame to a frame moving parallel to the wire results in a cylindrically symmetric radial electric field centered on the wire. The electric field lines either originate on the wire and extend radially outward or converge radially inward and terminate on the wire, implying a net charge density on the wire in the moving frame. This conclusion is based on a misapplication of Gauss' law.

Consider the following thought experiment. Have a fleet of closely-spaced observers all moving parallel to a current-carrying wire at the same speed in such a way that each detects an electric field extending away from the wire in a radial direction. They would agree, upon getting together and comparing observations, that there was an outwardly directed radial electric field around the wire. The expectation then would be if one released a positive charge, it would accelerate along a radial line away from the wire.

But this is not what happens. As soon as the released charge begins to move in a radial direction, it curves in the direction behind the release point. Its detection by the observers will show it to move in a looped pattern in the same direction the wire is moving in their reference frame. The initial motion of the charge is as expected, but then it moves in such a way that the observers may be puzzled. If they understand special relativity and that the frame of reference of the charge was no longer the same as theirs once it began moving, they might suspect their conclusion of a charge on the wire in their frame is incorrect.

The idea that the radial electric field lines resulting from a Lorentz transformation of the wire's magnetic field is due to a net charge on the wire from differential Lorentz contraction also runs into difficulty when applied to current loops. For a moving current loop it is supposed that one side of the loop is positive and the other negative. In that case the electric field lines should all originate on the positive side of the loop and terminate on the negative side.

However, this is not true of the electric field lines of the Lorentz-transformed magnetic field of the loop. For example, the transformed field lines in the plane of and outside the loop are all perpendicular to the direction of the loop's motion and extend to positive infinity, negative infinity, or to both (for field lines that don't intercept the loop). This would imply that there is charge at an infinite distance from the loop were these lines due to an electrostatic field. Even worse, there is no electric field parallel to the direction of motion of the loop at any point in space – a very un-dipole-like pattern.

There are two ways an electric field can be generated. An electrostatic field originates on positive charges and terminates on negative charges and is due to Gauss' law. What I call a Faraday field arises from a magnetic field changing with time (Faraday's law) and neither originates nor terminates on electric charges. In addition, how an electromagnetic field is perceived by an observer depends on the observer's velocity by the Lorentz transformation. (It will be shown below that the Lorentz transformation and Faraday's law can be combined into one equation.) The idea that the electric field of a moving magnetic dipole involves a charge separation on the dipole is incompatible with the actual field lines. I will discuss this in greater detail later on when these fields are derived.

The appearance of an electric dipole on a moving magnetic dipole (or a magnetic dipole on a moving electric dipole) is inferred from applying the Lorentz transformation to the magnetization-polarization tensor [3]. However, as was pointed out above and is well known, calculating the transformed electromagnetic fields assuming they are due to these induced dipoles does not give the same fields that result from a Lorentz transformation of the fields of the moving dipoles themselves. This leads to questions as to what algebraic expression resulting from the multipole expansions of fields is "the" induced dipole [4, 5]. In this paper I address these problems and argue that you cannot Lorentz-transform dipoles from one reference frame to another as you can fields. Actually, there is no "the" dipole.

II. CHARGE SEPARATION ON A CURRENT LOOP VERSUS TRANSFORMED FIELDS

The theoretical basis for differential Lorentz contraction was given most clearly, in my opinion, by Gabuzda [1]. In this paper she argued that a current carrying wire between a source and sink of charge was neutral in the rest frame of the wire despite the expected increase in linear electron density due to Lorentz contraction of the average electron spacing. An observer stationary with respect to the wire would detect that the electrons were closer together due to their drift speed if this were the only effect on the electron density.

However, the source and sink of charge are also in the rest frame of the wire and would see the wire as charged. Gabuzda argued that the excess electrons would move into the electron sink, making the wire neutral in its rest frame. Then, transforming the system of wire plus source and sink to the frame of an observer moving with speed v parallel to the wire, she showed that the fields seen by the observer would be the same as those calculated from a Lorentz transformation of the fields from the rest frame of the wire to that of the observer.

As for current loops without source or sink, such as created by changing magnetic flux, Gabuzda notes that the loop as a whole will remain neutral in spite of Lorentz contraction and so a test charge at rest with the loop would not experience a force. She does not consider the case where the loop and charge are in relative motion. Unless the charge is along the symmetry axis of the loop and the loop is moving along that axis, there will be an electric field present at the position of the charge due to the Lorentz-transformed magnetic field of the loop to the charge's frame of reference. The differential Lorentz contraction point of view is that one side of the loop is positively charged and the other side negative in the laboratory frame. This view led Mansuripur [6] to propose the paradox named after him.

In the Mansuripur paradox there is a charge in the vicinity of a current loop and at rest in the loop's reference frame. To simplify the problem the charge is taken to be in the plane of (but outside) the loop. One can imagine the charge is at the origin of a Cartesian coordinate system and the magnetic moment of the current loop is on the positive x axis with the dipole moment oriented in the positive z direction. The Lorentz transformation of a stationary magnetic dipole to a moving frame of reference is thought to result in an electric dipole centered on the magnetic dipole. In the frame of reference of an observer moving toward the charge-loop system in the positive x direction, the electric dipole moment is oriented perpendicular to the magnetic dipole in the positive y direction.

According to the idea of differential Lorentz contraction, the moving observer sees the left side of the loop positive (in the positive y direction) and the right side negative. Hence it is supposed she must see that the loop experiences a torque due to the action of the field of the point charge on the charge separation, a torque not seen by an at-rest observer. These two different observations are obviously contradictory. Mansuripur [6] claimed the contradiction can be resolved if the Lorentz force responsible for the torque were replaced by a force proposed by Einstein and Laub [7].

There are significant problems with the idea there is a charge separation on the current loop. An electric dipole due to a charge separation as described above would produce an electric field in the x-y plane that would consist of loops originating on the side of the loop with the positive charge and terminating on the side of the loop with the negative charge. An observer moving as indicated above should see an electric field pointing in the negative y direction if this were the case.

However, applying the right-hand rule to her situation, it is clear she sees the electric field in the *positive* y direction. In fact the electric field in the x-y plane is in the positive y direction *everywhere* outside the loop for observers moving as she is! In addition there is no electric field parallel to her motion wherever she might be. This is not the field of an electric dipole due to a charge separation. It is an error to consider the "electric dipole" that results from a Lorentz transformation of a magnetic dipole to be the same as that of an electric dipole with an actual charge separation. This has been noted in the literature and work has been done to explain the fields of moving magnetic and electric dipoles. For example, see [4, 5].

The resolution of the Mansuripur paradox has been thought by many of those supporting the Lorentz force to be due to the presence of hidden momentum in the charge-magnetic loop system [8–12]. I have shown [13] that the actual resolution has to do with the transformation of the angular electromagnetic momentum in the charge-magnetic loop system [14] from its rest frame to the lab frame in which it is moving. The torque that results from this transformation cancels that due to the charge-"electric dipole" interaction. This resolution also preserves the Lorentz force.

The torque that is supposed to be acting on the loop due to the presence of the point charge is not due to the interaction between the point charge and a charge separation on the loop. Instead, this torque exists as a time component of the torque four-vector in the rest frame of the charge-loop system. The charge-loop system also contains linear and angular electromagnetic momentum in the electromagnetic field [14]. The linear momentum is expressed in a time component of the angular momentum four-tensor. When these four-tensors are Lorentz transformed to the lab frame, the torques that appear as space components cancel [13]. No hidden momentum is necessary, therefore, to resolve this paradox. (In reference [13] I referred to the torques as "fictitious". This was a poor choice of words as the torques are not fictitious but originate in the electromagnetic field.)

III. LORENTZ TRANSFORMATION OF THE CURRENT DENSITY OF A WIRE

The relativistic treatment of the fields around a current-carrying wire (a straight segment – possibly very short – in a current loop with no source or sink) involves the Lorentz transformation of Maxwell's equations. The field-strength tensor [15], which is the covariant form of the electric and magnetic fields, is given (in SI units) by

$$(F^{\mu\nu}) = \begin{pmatrix} 0 & B_3 & -B_2 & -\frac{E_1}{c} \\ -B_3 & 0 & B_1 & -\frac{E_2}{c} \\ B_2 & -B_1 & 0 & -\frac{E_3}{c} \\ \frac{E_1}{c} & \frac{E_2}{c} & \frac{E_3}{c} & 0 \end{pmatrix} \tag{1}$$

where the coordinates are $x_1 = x$, $x_2 = y$, $x_3 = z$, and $x_4 = ct$.

Let there be a current density J' in a primed frame moving in the laboratory frame (unprimed) in the $x_{3'}$ direction. (This will later become the current I' in a wire lying along the $x_{3'}$ axis and moving parallel to it.) The wire is neutrally charged so that the electric field in the primed frame is zero. (This would be due to electrostatic forces between the charge carriers and fixed ions.) Also there is no magnetic field in the $x_{3'}$ direction. The field-strength tensor becomes

$$(F^{\mu'\nu'}) = \begin{pmatrix} 0 & 0 & -B_{2'} & 0 \\ 0 & 0 & B_{1'} & 0 \\ B_{2'} & -B_{1'} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2}$$

The inhomogeneous Maxwell equations are written in covariant form [16] as

$$\sum_{\nu'} \frac{\partial F^{\mu'\nu'}}{\partial x^{\nu'}} = \mu_o J^{\mu'}, \tag{3}$$

in SI units, where

$$J^{\mu'} = (\boldsymbol{J}', c\rho') = (J_{1'}, J_{2'}, J_{3'}, c\rho'), \tag{4}$$

in general, and where $\boldsymbol{J}'$ is the current density and ρ' is the charge density, and, for this case,

$$J^{\mu'} = (0, 0, J', 0). \tag{5}$$

To transform these Maxwell equations from the rest (unprimed) frame a moving (primed) frame parallel to the wire (in the x_3 direction), you must apply the Lorentz transformation [17] to Eq. (3). In matrix form you have

$$\left(\sum_{\nu} \frac{\partial F^{\mu\nu}}{\partial x^{\nu}}\right) = \begin{pmatrix} \frac{\partial B_3}{\partial x_2} - \frac{\partial B_2}{\partial x_3} - \frac{1}{c^2}\frac{\partial E_1}{\partial t} \\ -\frac{\partial B_3}{\partial x_1} + \frac{\partial B_1}{\partial x_3} - \frac{1}{c^2}\frac{\partial E_2}{\partial t} \\ \frac{\partial B_2}{\partial x_1} - \frac{\partial B_1}{\partial x_2} - \frac{1}{c^2}\frac{\partial E_3}{\partial t} \\ \frac{1}{c}\frac{\partial E_1}{\partial x_1} + \frac{1}{c}\frac{\partial E_2}{\partial x_2} + \frac{1}{c}\frac{\partial E_3}{\partial x_3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma_v & \gamma_v\beta \\ 0 & 0 & \gamma_v\beta & \gamma_v \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \mu_o J' \\ 0 \end{pmatrix} \tag{6}$$

where $\beta = v/c$. The individual equations of (6) are the inhomogeneous Maxwell equations in the lab frame. They are

$$\frac{\partial B_3}{\partial x_2} - \frac{\partial B_2}{\partial x_3} - \frac{1}{c^2}\frac{\partial E_1}{\partial t} = 0 \tag{7}$$

$$-\frac{\partial B_3}{\partial x_1}+\frac{\partial B_1}{\partial x_3}-\frac{1}{c^2}\frac{\partial E_2}{\partial t}=0 \tag{8}$$

$$\frac{\partial B_2}{\partial x_1}-\frac{\partial B_1}{\partial x_2}-\frac{1}{c^2}\frac{\partial E_3}{\partial t}=\gamma_v\mu_o J' \tag{9}$$

$$\frac{1}{c}\left(\frac{\partial E_1}{\partial x_1}+\frac{\partial E_2}{\partial x_2}+\frac{\partial E_3}{\partial x_3}\right)=\frac{v}{c}\gamma_v\mu_o J' \tag{10}$$

In what may be more familiar form, these equations are

$$\boldsymbol{\nabla}\times\boldsymbol{B}-\frac{1}{c^2}\frac{\partial\boldsymbol{E}}{\partial t}=\gamma_v\mu_o\boldsymbol{J'}=\mu_o\boldsymbol{J}, \tag{11}$$

and

$$\boldsymbol{\nabla}\cdot\boldsymbol{E}=v\gamma_v\mu_o J'=v\mu_o J=\frac{\rho}{\epsilon_o}. \tag{12}$$

The current density is due to a narrow flow of charge along the x_3 axis. It can be expressed in Dirac delta functions as

$$J'=I'\delta(x_{1'})\delta(x_{2'})=\gamma_{u'}en'u'\delta(x_1)\delta(x_2), \tag{13}$$

where u' is the drift speed in the moving frame,

$$I'=\gamma_{u'}en'u', \tag{14}$$

and where $\gamma_{u'}$, is given by

$$\gamma_{u'}=\frac{1}{\sqrt{1-\dfrac{u'^2}{c^2}}}. \tag{15}$$

Also used is the fact that, for this Lorentz transformation, $x_1=x_{1'}$ and $x_2=x_{2'}$.

Equation (12) is Gauss' law. The right-hand side of this equation multiplied by ϵ_o is the charge density ρ in the lab frame. Is this a real charge density? It certainly appears to be as it makes the divergence of the Lorentz-transformed electric field nonzero, but this result is actually due to the relativity of simultaneity as will be shown.

To get an expression for the differential "charge density" on the wire, you integrate ρ over x_1 and x_2. The result is (using $\mu_o\epsilon_o=1/c^2$)

$$\begin{aligned} e\Delta n &= \int \rho dx_1 dx_2=\frac{v}{c^2}\gamma_v I'\int\delta(x_1)\delta(x_2)dx_1dx_2 \\ &= \frac{v}{c^2}\gamma_v I', \end{aligned} \tag{16}$$

or, using Eq. (14),

$$e\Delta n=\gamma_v\gamma_{u'}\frac{u'v}{c^2}en' \tag{17}$$

To finish the solution, you solve the Maxwell equations, noting that, by symmetry, $B_{3'}$ and $E_{3'}$ are zero as are derivatives with respect to $x_{3'}$. The well-known results are that the magnitude of the magnetic field turns out to be

$$B=\gamma_v\frac{\mu_o}{2\pi}\frac{I'}{r}, \tag{18}$$

with the electric field magnitude given by $E=vB$.

IV. THE APPEARANCE OF CHARGE AND THE RELATIVITY OF SIMULTANEITY

In this section I will argue that the apparent linear charge, Eq.(17), on a current-carrying wire is due to the relativity of simultaneity, not to a physical charge density. In Fig. 1 you see the (vertical) worldlines of the ends (A′ and B′) of a segment of a current-carrying wire in the spacetime diagram of the wire's rest frame, where the positive current is flowing in the negative x direction. These are also the worldlines of two observers on the wire. Additionally there are the worldlines of two stationary observers (the diagonal lines A and B as seen from the wire's frame) as the wire moves past their positions in the negative x direction. The line at A is also the ct axis of the stationary observers spacetime diagram – the unprimed laboratory frame).

The wire segment has a proper length of l, whereas the two end stationary observers are γl apart in their rest frame and l apart in the primed frame. The two dots on the horizontal line are events corresponding to when the at-rest and moving observers are abreast of each other at the same time in the primed (wire) frame.

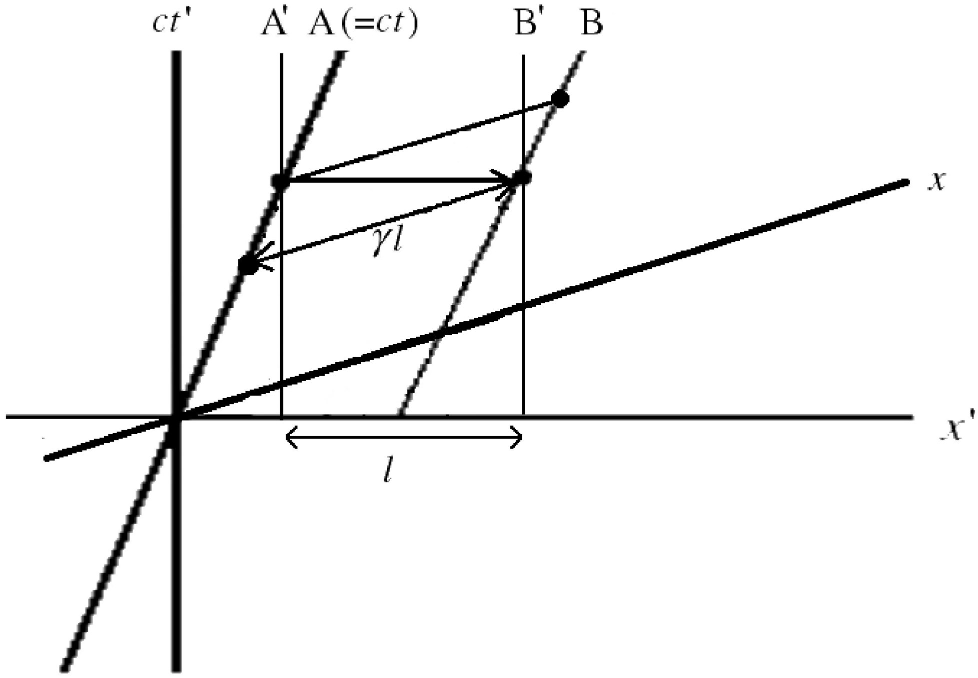

FIG. 1. Spacetime diagrams for at-rest and moving observers along an at-rest current-carrying wire. See text for explanation.

The distance between charges at A′ and B′ at the two events is l as seen in the frame of the moving wire. However, in the stationary laboratory frame the two events defining the separation of charges are not simultaneous like they are in the moving frame. When the left-moving observer B′ reaches stationary observer B, observer A′ has yet to reach observer A in the lab frame. By the time observer A is at the same position as A′, observer B is farther down the wire and a certain amount of time has elapsed in the lab frame frame since he was abreast of B′. In the stationary frame, B considers an amount of charge has traveled down the wire that observer A has yet to see. Hence there is an excess amount of charge on the wire in their reference frame at a given unprimed time t.

To quantify this, imagine a rectangular current loop with its length l' parallel to the x axis and moving in the positive x direction (to the right) with speed v in the lab (unprimed) frame (Fig. 2). A positive current I' is circulating in the rest frame of the loop in a counterclockwise direction such that it is flowing to the right in the lower side of the loop. There are n' positive charge carriers per unit length in the loop, each with a charge e (making the fixed ions negative). The average drift speed of the charge carriers is u' in the loop's rest frame, such that an observer moving along with the loop sees a positive current given by Eq. (14).

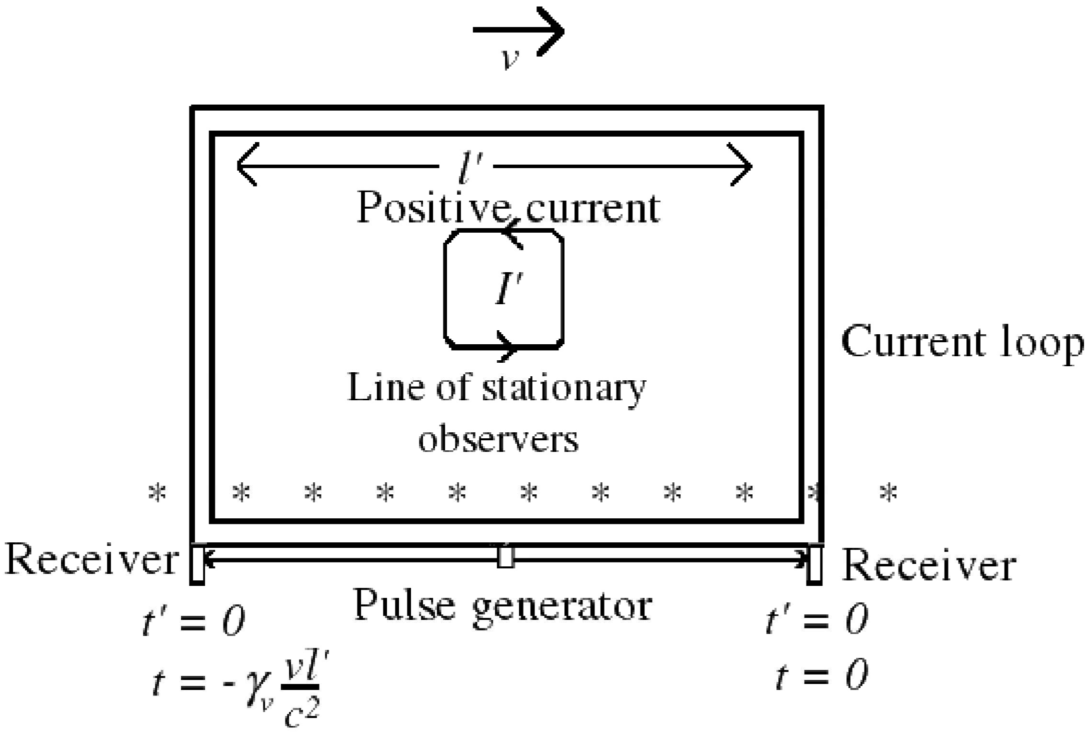

FIG. 2. What a observer (at a sufficient distance) would see when moving at the same speed v as the apparatus described in the text. (The primed values refer to the moving frame.)

In the lab frame the drift speed in the bottom wire due to relativistic velocity addition [18], would be

$$u = \frac{u' + v}{1 + \frac{u'v}{c^2}} - v = \frac{(c^2 - v^2)u'}{c^2 + u'v}, \tag{19}$$

and a current I given by

$$I = enu = e(n'\gamma_{u+v})\frac{(c^2 - v^2)u'}{c^2 + u'v} = \frac{\gamma_{u'}}{\gamma_v}en'u', \tag{20}$$

where

$$n = n'\gamma_{u+v} = n'\frac{1}{\sqrt{1 - \frac{(u+v)^2}{c^2}}} = n'\frac{c^2 + u'v}{\sqrt{(c^2 - v^2)(c^2 - u'^2)}} = n'\gamma_{u'}\gamma_v\frac{c^2 + u'v}{c^2}. \tag{21}$$

Now imagine that there are observers in the lab frame with synchronized clocks, lined up to observe the current in the lower wire of the loop as it passes each of their positions. Borrowing an example of the relativity of simultaneity from page 38 of Wolfgang Rindler's book, *Relativity: Special, General, and Cosmological* [19], imagine that there is a laser pulse generator located halfway along the bottom wire which projects identical pulses of sufficient briefness at the same time toward two receivers at each end of the wire. In the stationary frame a pulse approaches the leading receiver on the right at a relative speed of $c - v$ and the trailing receiver on the left at a relative speed of $c + v$. The distance the pulses travel will be the Lorentz-contracted distance $l'/2\gamma_v$. Hence, the observer who is abreast of the trailing receiver when the pulse arrives will record a time of arrival of

$$t = \frac{l'}{2\gamma_v(c - v)} - \frac{l'}{2\gamma_v(c + v)} = \gamma_v\frac{vl'}{c^2} \tag{22}$$

earlier than that of an observer at the leading receiver when the pulse arrives there. Therefore, by her estimation, an amount of charge given by

$$q = I(t-0) = \left(\frac{\gamma_{u'}}{\gamma_v} en'u'\right)\left(\gamma_v \frac{vl'}{c^2}\right) = \gamma_{u'} \frac{u'v}{c^2} en'l' \quad (23)$$

has flowed past her position before the observer at the leading receiver sees the arrival of the pulse, using Eq. (20) for I. That is, the observer at the leading receiver has not seen this charge pass his position when the pulse arrives.

The stationary observers must conclude, upon comparing notes, that there was an excess charge of q on the wire in their reference frame between pulse arrivals compared to that seen in the moving frame between the same two events. They therefore expect there to be an excess linear charge density on the wire given by

$$e\Delta n = \frac{q}{l'/\gamma_v} = \gamma_v \gamma_{u'} \frac{u'v}{c^2} en' \quad (24)$$

in their reference frame, the same as that found in Eq. (17) as a result of performing a Lorentz transformation on the current-density four-vector. On the other hand they realize that this extra charge was not present in the traveling frame *and was only seen in their frame due to the effect of the relativity of simultaneity.*

If this "charge density" is used to calculate an electric field, the result is

$$E = \frac{1}{2\pi\epsilon_o} \frac{\gamma_v \gamma_{u'} \frac{u'v}{c^2} en'}{r} = \gamma_v v B' \quad (25)$$

where B' is given by

$$B' = \frac{\mu_o}{2\pi} \frac{I'}{r}, \quad (26)$$

I' by Eq. (14), and $c^{-2} = \epsilon_o \mu_o$. Note that this result does not require an actual net charge to be present on the wire. When the observers compare notes, they see that the excess charge was the result of their measuring the charge flow at times that were not simultaneous with what was happening in the moving system, but this is not why they detect a radial field. That is due soley to the Lorentz transformation as will be discussed later.

Eq. (24) is the same excess charge density as that found by Gabuzda [1] arguing from the point of differential Lorentz contraction. Differential Lorentz contraction occurs because of the relativity of simultaneity in the following way. Because of timing differences between the moving and stationary reference frames, the stationary observer at the trailing end of the current loop sees an excess charge move onto the lower wire of the loop that the leading observer does not see when they make observations at the same time in their reference frame. Hence, at identical times in their reference frame there are more charge carriers on the lower wire than there are at identical times in the moving frame, hence the charge carriers appear to be closer together. Yet, once again, it is the relativity of simultaneity that produces this effect.

Unfortunately, if the lab-frame observers conclude that the effect of the electric field of a moving wire on the motion of a charge is globally radial, they are incorrect as has been shown. The electric field solution found above from the apparent charge density at the location of the observer cannot be extended beyond the motion and position of that observer. Also incorrect is the conclusion that the electric field of a moving loop is due to the inferred charge distribution of either the relativity of simultaneity or differential Lorentz contraction. The electric field due to the transformed magnetic field of the loop is not compatible with these scenarios as will be shown below.

The problem is the motion of the current loop. Calculating the dipole moment as if there were an actual charge separation in a frame where the loop is stationary ignores relativistic timing differences. It should be clearly impermissible to calculate the dipole moment with the inferred charge distribution of the moving loop assuming the loop is stationary. The Lorentz transformation of the magnetic field itself is not subject to this caveat, since the field results from the effect of the current at the *same time* in the reference frame stationary with respect to the loop (ignoring retardation effects).

Consider this thought experiment. Replace the bottom wire of the loop with a wire stretching between two equally and oppositely charged plates. In the rest frame of this apparatus a positive current flows from left to right in the wire from the positive to the negative plate. The sum of the charges on the plates is zero and the wire is neutal. Now look at the situation in a reference frame in which this apparatus is moving from left to right. According to the idea of differential Lorentz contraction, the wire is positively charged. This means the charges on the plates are less positive than in the rest frame. There is no Lorentz contraction effect for the plates, so how is it the two frames of observation disagree on the amount of charge on the plates? This appears to be a violation of charge conservation.

Also consider the following. Say you have a Gilbertian magnet consisting of a gap between two poles of magnetic charges creating a magnetic field $\boldsymbol{B}$ in the gap. An electric charge inserted into this gap with a velocity $\boldsymbol{v}$ would experience a Lorentz force. The Lorentz transformed electric field in the reference frame of the charge would be $\gamma\boldsymbol{v}\times\boldsymbol{B}$. However, there can be no charge separation due to differential Lorentz contraction or inferred charge due to the relativity of simultaneity here. The Lorentz transformation applied to electromagnetic fields is local. In my opinion the nature of the sources of the fields has nothing to do with this. Whatever the source of a magnetic field, be it Amperian or Gilbertian, the Lorentz transformation works the same way, meaning the supposed net charge on a moving wire is not responsible for the electric field. In the next section I will show that the electric field due to the Lorentz transformation of the magnetic field of a dipole is not a dipole field.

V. THE TRANSFORMED FIELDS OF A MOVING MAGNETIC DIPOLE

The magnetic field of a point magnetic dipole located at the origin of a coordinate system in its rest (primed) frame is

$$\boldsymbol{B}' = \frac{\mu_o}{4\pi}\left[\frac{3(\boldsymbol{m}'\cdot\boldsymbol{r}')\boldsymbol{r}'}{r'^5} - \frac{\boldsymbol{m}'}{r'^3}\right] - \frac{2\mu_o\boldsymbol{m}'}{3}\delta(\boldsymbol{r}'), \tag{27}$$

where $\boldsymbol{r}' = x'\hat{\boldsymbol{\imath}} + y'\hat{\boldsymbol{\jmath}} + z'\hat{\boldsymbol{k}}$ and $\delta(\boldsymbol{r}')$ is the Dirac delta function. (The delta function accounts for the singularity at the location of the dipole. It can be ignored for this study.) This is also the magnetic field of a current loop at a distance from the loop large compared to the loop's radius. Now have the primed frame moving in the positive x direction in the unprimed laboratory frame with speed v. When the origin of the lab frame coincides with that of the primed frame, you find

$$\boldsymbol{B} = B_{x'}\hat{\boldsymbol{\imath}} + \gamma B_{y'}\hat{\boldsymbol{\jmath}} + \gamma B_{z'}\hat{\boldsymbol{k}} \tag{28}$$

and

$$\boldsymbol{E} = -\boldsymbol{v}\times\boldsymbol{B} = \gamma v B_{z'}\hat{\boldsymbol{\jmath}} - \gamma v B_{y'}\hat{\boldsymbol{k}}, \tag{29}$$

the fields in the laboratory frame. The position vector in the primed frame transforms to $\boldsymbol{r}' = \gamma x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}$ so that $r'^2 = \gamma^2 r^2 - (\gamma^2-1)(y^2+z^2) = \gamma^2 r^2(1-(v^2/c^2)sin^2\alpha)$, where $\boldsymbol{r} = x\hat{\boldsymbol{k}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}$ is the position vector in the unprimed frame and α is the angle between $\boldsymbol{v}$ and $\boldsymbol{r}$ [20].

Taking $\boldsymbol{m}'$ to be in the positive z direction, the components of the magnetic field in the primed frame are

$$\begin{aligned} B_{x'} &= \frac{3\mu_o m'}{4\pi}\frac{x'z'}{r'^5}, \\ B_{y'} &= \frac{3\mu_o m'}{4\pi}\frac{y'z'}{r'^5}, \\ B_{z'} &= \frac{\mu_o m'}{4\pi}\left[\frac{3z'^2}{r'^5} - \frac{1}{r'^3}\right]. \end{aligned} \tag{30}$$

The transformed field components are,

$$\begin{aligned} B_x &= \frac{\mu_o m'}{4\pi}\frac{3xz}{\gamma^4 r^5(1-(v^2/c^2)sin^2\alpha)^{5/2}}, \\ B_y &= \frac{\mu_o m'}{4\pi}\frac{3yz}{\gamma^4 r^5(1-(v^2/c^2)sin^2\alpha)^{5/2}}, \\ B_z &= \frac{\mu_o m'}{4\pi}\left[\frac{3z^2}{\gamma^4 r^5(1-(v^2/c^2)sin^2\alpha)^{5/2}} - \frac{1}{\gamma^2 r^3(1-(v^2/c^2)sin^2\alpha)^{3/2}}\right]. \end{aligned} \tag{31}$$

and

$$\begin{aligned} E_x &= 0, \\ E_y &= \frac{\mu_o v m'}{4\pi}\left[\frac{3z^2}{\gamma^4 r^5(1-(v^2/c^2)sin^2\alpha)^{5/2}} - \frac{1}{\gamma^2 r^3(1-(v^2/c^2)sin^2\alpha)^{3/2}}\right]. \\ E_z &= -\frac{\mu_o v m'}{4\pi}\frac{3yz}{\gamma^4 r^5(1-(v^2/c^2)sin^2\alpha)^{5/2}}, \end{aligned} \tag{32}$$

Note that $(\hat{\boldsymbol{k}}\cdot\boldsymbol{r})\boldsymbol{r} = xz\hat{\boldsymbol{\imath}} + yz\hat{\boldsymbol{\jmath}} + z^2\hat{\boldsymbol{k}}$. This means you can express the equations in (31) in a coordinate-free way as

$$\boldsymbol{B} = \frac{\mu_o}{4\pi}\left[\frac{3(\gamma\boldsymbol{m}'\cdot\boldsymbol{r})\boldsymbol{r}}{\gamma^5 r^5[1-(v^2/c^2)sin^2\alpha]^{5/2}} - \frac{\gamma\boldsymbol{m}'}{\gamma^3 r^3[1-(v^2/c^2)sin^2\alpha]^{3/2}}\right]. \tag{33}$$

Now $1-(v^2/c^2)sin^2\alpha = \gamma^{-2} + (v^2/c^2)cos^2\alpha$ where $cos\alpha$ is the direction cosine from the $\boldsymbol{v}$ direction. This allows the equation above to be written as

$$\boldsymbol{B} = \frac{\mu_o}{4\pi}\left[\frac{3(\gamma\boldsymbol{m}'\cdot\boldsymbol{r})\boldsymbol{r}}{r^5[1+\gamma^2(v^2/c^2)cos^2\alpha]^{5/2}} - \frac{\gamma\boldsymbol{m}'}{r^3[1+\gamma^2(v^2/c^2)cos^2\alpha]^{3/2}}\right]. \tag{34}$$

In the slow-motion case ($v << c$) you can set the quantities in the brackets in the denominators equal to one, such that the transformed dipole is $\boldsymbol{m} = \gamma\boldsymbol{m}'$. However, little appears to be gained by this since γ is taken to be one in the slow-motion case anyway.

The real problem is with the electric field resulting from the transformation, $\boldsymbol{E} = -\boldsymbol{v}\times\boldsymbol{B}$. The electric field at the moment the axes of the laboratory and primed frames are aligned is

$$\boldsymbol{E} = \frac{\mu_o}{4\pi}\left[-\frac{3\gamma(\boldsymbol{m}'\cdot\boldsymbol{r})\boldsymbol{v}\times\boldsymbol{r}}{\gamma^5 r^5[1-(v^2/c^2)sin^2\alpha]^{5/2}} + \frac{\gamma\boldsymbol{v}\times\boldsymbol{m}'}{\gamma^3 r^3[1-(v^2/c^2)sin^2\alpha]^{3/2}}\right]. \tag{35}$$

The usual definition of the electric dipole resulting from the motion of a magnetic dipole is $\boldsymbol{p} = \gamma\boldsymbol{v}\times\boldsymbol{m}'/c^2$ such that there should be a term $3\gamma(\boldsymbol{v}\times\boldsymbol{m}'\cdot\boldsymbol{r})\boldsymbol{r}/r^5$ inside the brackets. To introduce $\boldsymbol{p}$, the numerator of the first term in brackets can be written using a vector identity as $(\boldsymbol{m}'\cdot\boldsymbol{r})(\boldsymbol{v}\times\boldsymbol{r}) = -(\boldsymbol{v}\times\boldsymbol{m}'\cdot\boldsymbol{r})\boldsymbol{r} + (\boldsymbol{v}\times\boldsymbol{m}')r^2 - (\boldsymbol{r}\times\boldsymbol{m}')(\boldsymbol{v}\cdot\boldsymbol{r})$. Substituting this and the expression for $\boldsymbol{p}$ in the above equation results in

$$\boldsymbol{E} = \frac{1}{4\pi\epsilon_o}\left[\frac{3(\boldsymbol{p}\cdot\boldsymbol{r})\boldsymbol{r} - 3\boldsymbol{p}r^2 + 3(\boldsymbol{r}\times\boldsymbol{m}')(\boldsymbol{v}\cdot\boldsymbol{r}/c^2)}{\gamma^5 r^5[1-(v^2/c^2)sin^2\alpha]^{5/2}} + \frac{\boldsymbol{p}}{\gamma^3 r^3[1-(v^2/c^2)sin^2\alpha]^{3/2}}\right]. \tag{36}$$

For the slow-motion case, this equation can be expressed as

$$\boldsymbol{E} = \frac{1}{4\pi\epsilon_o}\left[\frac{3(\boldsymbol{p}\cdot\boldsymbol{r})\boldsymbol{r}}{r^5} - \frac{\boldsymbol{p}}{r^3} + \frac{3(\boldsymbol{r}\times\boldsymbol{m}')(\boldsymbol{v}\cdot\boldsymbol{r}/c^2)}{r^5} - \frac{\boldsymbol{p}}{r^3}\right]. \tag{37}$$

The first two terms in brackets are the terms of a point electric dipole, and if those were the only terms, you would definitely have an electric dipole present on a moving magnetic dipole. Unfortunately, you also have the last two terms. These are also dipole-like but alter the pure dipole terms such that the equation is not that of a point electric dipole. In particular, there is no electric field parallel to the $\boldsymbol{v}$ direction.

VI. THE TRANSFORMED FIELDS OF A MOVING ELECTRIC DIPOLE

The electric field of an electric charge q moving with velocity $\boldsymbol{v}$ is given by [20]

$$\boldsymbol{E} = \frac{1}{4\pi\epsilon_o}\frac{q\boldsymbol{r}}{\gamma^2 r^3(1-\beta^2 sin^2\alpha)^{3/2}}, \tag{38}$$

when the charge is at the origin, and where $\beta = v/c$ and the other symbols are defined as previously. The field due to two adjacent opposite charges separated by a vector $2\boldsymbol{a}$ pointing from the negative to the positive charge is

$$\boldsymbol{E} = \frac{q}{\gamma^2 4\pi\epsilon_o}\left[\frac{\boldsymbol{r}-\boldsymbol{a}}{|\boldsymbol{r}-\boldsymbol{a}|^3}(1-\beta^2 sin^2\alpha_+)^{-3/2} - \frac{\boldsymbol{r}+\boldsymbol{a}}{|\boldsymbol{r}+\boldsymbol{a}|^3}(1-\beta^2 sin^2\alpha_-)^{-3/2}\right], \tag{39}$$

where the origin of the coordinate system (origin of $\boldsymbol{r}$) is halfway between the two charges; $\boldsymbol{r}-\boldsymbol{a}$ and $\boldsymbol{r}+\boldsymbol{a}$ are the field positions with respect to charges $+q$ and $-q$, respectively; and α_+ and α_- are the angles between $\boldsymbol{v}$ and $\boldsymbol{r}-\boldsymbol{a}$ and $\boldsymbol{r}+\boldsymbol{a}$, respectively. All quantities in the above two equations are defined in the laboratory reference frame. (See the figure.)

In finding the dipole field as the dipole moment is formed, the following approximations are used.

$$|\boldsymbol{r}\pm\boldsymbol{a}|^{-n} \approx \frac{1}{r^n}\left(1\mp 2n\frac{\boldsymbol{a}\cdot\boldsymbol{r}}{r^2}\right). \tag{40}$$

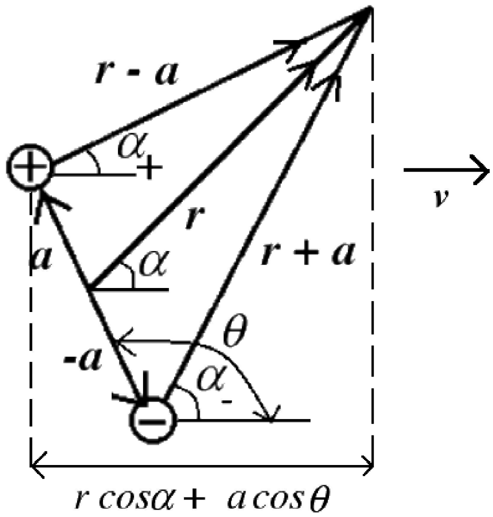

FIG. 3. Definition of geometrical terms for moving electric dipole as seen in laboratory frame.

$$sin^2\alpha_{\pm} = 1 - \frac{r^2cos^2\alpha + 2(ar)cos\alpha cos\theta}{|\boldsymbol{r} \mp \boldsymbol{a}|}$$
$$\approx sin^2\alpha \mp \frac{2\boldsymbol{a}\cdot\boldsymbol{r}}{r^2}cos^2\alpha + \frac{2(ar)cos\alpha cos\theta}{r^2}. \quad (41)$$

Here, α is the angle between $\boldsymbol{v}$ and $\boldsymbol{r}$, and θ is the angle between $\boldsymbol{v}$ and $2\boldsymbol{a}$. The above equation results from the fact that

$$|\boldsymbol{r}+\boldsymbol{a}|cos\alpha_- + 2acos\theta = rcos\alpha + acos\theta = |\boldsymbol{r}-\boldsymbol{a}|cos\alpha_+. \quad (42)$$

It should be kept in mind that the vector $2\boldsymbol{a}$ is Lorentz-contracted in the $\boldsymbol{v}$ direction such that $\boldsymbol{a} = \boldsymbol{a}'_{||}/\gamma + \boldsymbol{a}'_{\perp}$, where $\boldsymbol{p}' = 2q\boldsymbol{a}' = 2q(\boldsymbol{a}'_{||} + \boldsymbol{a}'_{\perp})$ in the (primed) frame stationary with respect to the dipole. In the laboratory frame you might expect $\boldsymbol{p} = 2\boldsymbol{q}a$. With the approximations adopted, Eq. (39) becomes

$$\boldsymbol{E} = \frac{1}{4\pi\epsilon_o\gamma^2}\left[\left(\frac{3(\boldsymbol{p}\cdot\boldsymbol{r})\boldsymbol{r}}{r^5} - \frac{\boldsymbol{p}}{r^3}\right)(1-\beta^2sin^2\alpha)^{-3/2} - 3\beta^2\frac{(\boldsymbol{p}\cdot\boldsymbol{r})\boldsymbol{r}cos^2\alpha}{r^5}(1-\beta^2sin^2\alpha)^{-5/2}\right]. \quad (43)$$

Eq. (43) is a dipole-like equation where the first term in brackets is the electric field of a dipole with a dipole moment given by $(1-\beta^2sin^2\alpha)^{-3/2}\boldsymbol{p}/\gamma^2$. However, the field as a whole is not strictly that of a dipole unless you invoke the slow-motion case where $\gamma = 1$ and $\beta = 0$. But once again the real problem is the induced field – a magnetic field this time. The magnetic field is given by

$$\boldsymbol{B} = \frac{1}{c^2}\boldsymbol{v}\times\boldsymbol{E} \approx \frac{\mu_o}{4\pi}\left[\frac{3(\boldsymbol{p}\cdot\boldsymbol{r})(\boldsymbol{v}\times\boldsymbol{r})}{r^5} - \frac{\boldsymbol{v}\times\boldsymbol{p}}{r^3}\right], \quad (44)$$

where the slow-motion version is displayed. Following the procedure used for Eq. (37), the magnetic field can be written as

$$\boldsymbol{B} = \frac{\mu_o}{4\pi}\left[\frac{3(\boldsymbol{m}\cdot\boldsymbol{r})\boldsymbol{r} - 3\boldsymbol{m}r^2 + 3(\boldsymbol{r}\times\boldsymbol{p})(\boldsymbol{v}\cdot\boldsymbol{r})}{r^5} + \frac{\boldsymbol{m}}{r^3}\right], \quad (45)$$

where $\boldsymbol{m} = \gamma\boldsymbol{p}\times\boldsymbol{v} \approx \boldsymbol{p}\times\boldsymbol{v}$ is the usual form used for the magnetic dipole of the transformation. This equation can be rewritten as follows.

$$\boldsymbol{B} = \frac{\mu_o}{4\pi}\left[\frac{3(\boldsymbol{m}\cdot\boldsymbol{r})\boldsymbol{r}}{r^5} - \frac{\boldsymbol{m}}{r^3} + \frac{3(\boldsymbol{r}\times\boldsymbol{p})(\boldsymbol{v}\cdot\boldsymbol{r})}{r^5} - \frac{\boldsymbol{m}}{r^3}\right]. \quad (46)$$

The first two terms in brackets do indeed constitute the field of a point magnetic dipole, but the last two terms modify this considerably. And, like Eq. (37) there is no magnetic field parallel to the $\boldsymbol{v}$ direction. Hence, both Eqs. (37) and (46) show that transforming the field of a moving dipole does not result in a transformed dipole of the other field.

VII. WHAT IS GOING ON?

Look again at the situation of an observer moving parallel to a current-carrying wire. Assume his motion and the direction of the current is such that he sees an electric field directed away from the wire as if there were a net positive charge on the wire. A whole fleet of such observers in the vicinity of the wire moving identically to the initial observer would all see an electric field directed away from the wire also. Combining these observations into an electric field is what I refer to as a "global" Lorentz transformation. But is this global result the correct picture of the electric field? Does this prove there must be a net positive charge on the wire? Certainly, the motion of a charge released by one of the observers indicates not.

Well, consider a magnet of the kind used in experiments where a large magnetic field is necessary. If an observer is injected into the magnet gap, moving parallel to the magnet faces, she would detect an electric field perpendicular to and to the left of her velocity. A fleet of spatially spread-out observers moving identically would detect an electric field directed from their right to their left. So, there must be positive charge outside the magnet to the right and negative charge to the left, correct? Of course not. In fact, when an observer moves out of the magnetic field, he sees the electric field go to zero.

Rather, to find the electric field lines you inject a positive test charge between the poles of the magnet moving like our observer. As a result of the Lorentz force, the charge will execute cyclotron motion. In the absence of friction and radiation, it will move in a circle, repeating this orbit over and over. If it has a velocity component parallel to the magnetic field, it will move in a spiral. The electric field line it follows neither originates on a positive charge nor terminates on a negative one but depends on the velocity of the charge. This is what I call a Faraday field in contrast to a Gauss (electrostatic) field with field lines anchored to electric charges. Because of the conservation of electric charge, once the field lines are defined the number of lines do not change during a Lorentz transformation of a Gauss field.

Due to the Lorentz transformation, an observer moving in a magnetic field $\boldsymbol{B}$ with velocity $\boldsymbol{v}$ can detect an electric field at his position given by

$$\boldsymbol{E} = \boldsymbol{v} \times \boldsymbol{B}. \tag{47}$$

Taking the curl of both sides of the above equation, using a common vector identity, and noting that $\nabla \cdot \boldsymbol{B} = 0$ results in

$$\nabla \times \boldsymbol{E} = -(\nabla \cdot \boldsymbol{v})\boldsymbol{B}, \tag{48}$$

if the magnetic field is not an explicit function of time. Combining this with Faraday's law gives you the formula

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} - (\nabla \cdot \boldsymbol{v})\boldsymbol{B} = -\frac{d\boldsymbol{B}}{dt}, \tag{49}$$

for an observer moving through a magnetic field changing with time. (A similar argument can be used to show that

$$\nabla \times \boldsymbol{B} = \mu_o \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} + (\nabla \cdot \boldsymbol{v})\boldsymbol{D} = \mu_o \boldsymbol{J} + \frac{d\boldsymbol{D}}{dt}, \tag{50}$$

where $\boldsymbol{D}$ is the electrical displacement and $\boldsymbol{J}$ is the current density in the moving frame.) The electric field arising from the Lorentz transformation is therefore like a Faraday field in that it neither originates on positive charges nor terminates on negative charges.

A charge injected parallel to a current-carrying wire will execute orbital motion, closed or open, similar to cyclotron motion. The Lorentz transformation is local such that a "global" transformation is misleading. You can't connect the field line detected by one observer in the fleet to that of another and then use Gauss' law to find the charge. This is a misapplication of that law due to ignoring relativistic time effects. The calculations of the fields of moving dipoles given above are also global Lorentz transformations for a given velocity. An observer moving with a different velocity will see different fields, and a released charge will not follow the electric field lines resulting from a global transformation of the magnetic field of a magnetic dipole.

You saw above that the charge separation that seems to appear on a moving current loop can be attributed to the relativity of simultaneity, yet the electric field of the moving loop is not that of an electric dipole. It is like a Faraday field, neither originating on a positive charge nor terminating on a negative charge. The electric field of a moving Gilbertian magnetic dipole will produce the same electric field as that found at a suitable distance from a current loop. There certainly can be neither differential Lorentz contraction nor the effect of the relativity of simultaneity here. Once again, you have an electric field consisting of closed or open-ended field lines and which depend on the relative motion between the source of the magnetic field and the point of observation. The same analysis applies to electric dipoles and Gilbertian magnetic current loops.

VIII. THE MAGNETIZATION-POLARIZATION TENSOR

In this section I am going to argue that you cannot use the transformation of the magnetization-polarization four-tensor to produce induced dipoles. Take, for example, something that is often done in the literature: transforming a magnetic dipole from its rest frame to a moving frame using this tensor, resulting in an electric dipole appearing on the magnetic dipole. From what I have presented above, it should be clear that the resulting "electric dipole" is not equivalent to an electric dipole formed by a charge separation. Using the tensor this way does not seem to be consistent with the definitions of magnetization and polarization, which are the average magnetic dipole moment and average electric dipole moment, respectively, per unit volume. The extra terms in Eqs. (37) and (46) must be used in calculations in addition to the dipole terms.

The magnetization-polarization tensor is defined indirectly as that tensor which, when differentiated and contracted, yields the effective current density four-vector due to magnetization and time-varying polarization [3]. That is,

$$J^{\mu}_{eff} = \frac{\partial M^{\mu\nu}}{\partial x^{\nu}} = (\nabla \times \boldsymbol{M} + \frac{\partial \boldsymbol{P}}{\partial t}, -c\nabla \cdot \boldsymbol{P}). \tag{51}$$

Here,

$$x^{\mu} = (x, y, z, ct), \tag{52}$$

and

$$(M^{\mu\nu}) = \begin{pmatrix} 0 & -M_z & M_y & cP_x \\ M_z & 0 & -M_x & cP_y \\ -M_y & M_x & 0 & cP_z \\ -cP_x & -cP_y & -cP_z & 0 \end{pmatrix}. \tag{53}$$

The polarization and magnetization are defined as, respectively, [21, 22]

$$\boldsymbol{P} = N < \boldsymbol{p}_{mol} >, \tag{54}$$

and

$$\boldsymbol{M} = N < \boldsymbol{m}_{mol} >, \tag{55}$$

where N is the number of molecules in a given region of a substance and $< \boldsymbol{p}_{mol} >$ and $< \boldsymbol{m}_{mol} >$ are the average electric and magnetic dipoles, respectively, per molecule in that region. The definition is made such that the region involved is much smaller than the scale of measurement and much larger than the size of an individual molecule. The terms $\nabla \times \boldsymbol{M}$ and $\partial \boldsymbol{P}/\partial t$ are the effective magnetization and polarization current densities, respectively, and $-\nabla \cdot \boldsymbol{P}$ is the effective charge density due to polarization. Considering these definitions, it is not clear that substituting individual dipoles for $\boldsymbol{M}$ and $\boldsymbol{P}$ is appropriate for this tensor.

However, there are also problems when the tensor is applied to bulk material. For example, consider a bulk sample of a substance with uniform and isotropic magnetization. It will be represented by a tensor with only magnetization components. When it is Lorentz-transformed to a frame in relative motion it will become a tensor with both magnetization and polarization components. If the tensor has a magnetization component in the positive z direction and the magnetic material is moving in the positive x direction, the transformed tensor will contain an electric polarization component directed in the negative y direction. At a great distance from this material it would be expected that the electric field would be like that of an electric dipole, but this will not be the same as the transformed electric field calculated directly from the moving magnetic field. In fact, the transformed fields of dipoles are identical to the fields expected if you apply Maxwell's equations. (This, of course, has to be the case.)

In a frame in which the dipoles are moving, the fields at any point in space are, in general, time-dependent. In the case of a moving magnetic dipole, Eq. (49) can be used to solve for the transformed electric field at the point of a moving particle. Applying the corrected version of Ampere's law to the time-dependent electric field of the moving electric dipole results in the transformed magnetic field.

A four-matrix appearing in an equation in special relativity is not a four-tensor if the equation does not remain valid when transformed from one inertial frame to another. In the frame of reference of the definition of the tensor, the fields are consistent with the polarization and magnetization of a bulk sample. Transform the tensor to another frame and this is no longer the case.

Whether you are talking about a moving current loop or Gilbertian magnetic dipole, there will be an electric field that has some characteristics of an electric dipole. However, this electric field is a Faraday field, not a Gauss field, and none of the field lines originate or terminate on electric charge. The relativity of simultaneity can explain the apparent presence of charge on a current loop but not on a Gilbertian magnetic dipole. Only fields transform in special relativity, not multipoles, and that is the basis of the problem with the magnetization-polarization tensor. I therefore argue that you cannot transform a Faraday-like field (magnetic or electric) into a field that originates or terminates on magnetic or electric charges.

IX. THE RELATIVITY OF SIMULTANEITY AND A MAGNETIZED BODY

To illustrate the role of the relativity of simultaneity for a magnetized body, I am going to examine a uniformly magnetized sphere in motion. If the magnetic moment of the sphere is $\boldsymbol{m}'$, the magnetization will be given by

$$\boldsymbol{M}' = \frac{3\boldsymbol{m}'}{4\pi R'^3}, \tag{56}$$

where R' is the radius of the sphere. This sphere can be modeled as a conducting sphere with a surface current density [14]

$$\boldsymbol{K}' = \frac{3}{4\pi R'^4}\boldsymbol{m}' \times \boldsymbol{R}' = \frac{3m'}{4\pi R'^3} sin\theta' \hat{\boldsymbol{\phi}}', \tag{57}$$

where $\boldsymbol{m}' = m'\hat{\boldsymbol{k}}$ and $\boldsymbol{R}' = R'(sin\theta' cos\phi' \hat{\boldsymbol{\imath}} + sin\theta' sin\phi' \hat{\boldsymbol{\jmath}} + cos\theta' \hat{\boldsymbol{k}})$. The choice of the direction of $\boldsymbol{m}'$ means the current is in the counterclockwise direction around the positive z axis. If the sphere is moving with speed v in the positive x direction, the transformation of the surface current density will result in what appears to be a surface charge density in the lab frame, given by

$$\sigma = \gamma \frac{v}{c^2} K_{x'} = -\gamma \frac{3vm'}{4\pi c^2 R'^3} sin\theta' sin\phi', \tag{58}$$

where $K_{x'}$ is the x-component of $\boldsymbol{K}'$. For the slow-motion case, $\gamma = 1$ and the primes can be dropped. An element of charge within the area $dS = R^2 sin\theta d\theta d\phi$ on the surface of the sphere is given by

$$dq = \sigma dS = -\frac{3vm}{4\pi c^2 R} sin^2\theta sin\phi d\theta d\phi. \tag{59}$$

There will thus appear to be a negative charge on the sphere from $\phi = 0$ to $\phi = \pi$ and a positive charge from $\phi = -\pi$ to $\phi = 0$. To find the apparent positive charge on the sphere, you integrate dq from $\phi = -\pi$ to 0 and $\theta = 0$ to π. The integral is

$$q_+ = -\frac{3vm}{4\pi c^2 R^2} \int_0^\pi sin^2\theta d\theta \int_{-\pi}^0 sin\phi d\phi = \frac{3vm}{4c^2 R}, \tag{60}$$

with, of course, an equal amount of negative charge on the opposite hemisphere. Note, however, that this calculation takes the charge to be that of the moving sphere but performs the calculation in the rest frame of the sphere. This is clearly incorrect.

Similarly, you might think you could integrate the quantity $\boldsymbol{R}dq$ over the surface of the sphere to get the electric dipole of the sphere. Doing this, you find

$$\boldsymbol{p} = \int_S \boldsymbol{R} dq = -\frac{3vm\hat{\boldsymbol{\jmath}}}{4\pi c^2} \int_0^\pi sin^3\theta d\theta \int_0^{2\pi} sin^2\phi d\phi = \frac{\boldsymbol{v} \times \boldsymbol{m}}{c^2}, \tag{61}$$

where the integrals over ϕ in the x and z directions are zero. This is the result expected from employing the magnetization-polarization tensor to a magnetic dipole. Why, then, is the transformed electric field not that of an electric dipole? Again the problem is the sphere is moving, and you can't use the apparent charge density of the moving sphere in a calculation that takes the sphere to be at rest.

In fact, the surface charge density results from the same effect as that of the current loop examined earlier: the relativity of simultaneity. Look at the current flowing between two points on the sphere a distance $dl = -Rsin\theta sin\phi d\phi$ apart as measured in the positive x direction on the "positive" side of the sphere ($sin\phi \leq 0$). You can use a slightly modified version of Eq. (23) to calculate the observed difference in charge due to the relativity of simultaneity between observers on either side of dl.

$$dq = \frac{v}{c^2} dI dl = -\frac{v}{c^2} dI R sin\theta sin\phi d\phi, \tag{62}$$

where

$$dI = \boldsymbol{K} \cdot Rd\theta\hat{\boldsymbol{\phi}} = \frac{3m}{4\pi R^2} sin\theta d\theta, \tag{63}$$

such that

$$dq = -\frac{3vm}{4\pi c^2 R} sin^2\theta sin\phi d\theta d\phi, \tag{64}$$

is the same as Eq. (59).

It might be argued that the above development ignores the fact that a point magnetic dipole cannot exhibit the relativity of simultaneity because, after all, it is but a point. However, the total current on the sphere is easily found to be $I = 3m/2\pi R^2$ such that $m = 2\pi IR^2/3$. To form an Amperian point dipole, you must hold m constant while I goes to infinity and R goes to zero. Eq. (64) shows that the charge difference due to the relativity of simultaneity goes to infinity as R goes to zero so that what is true for a finite sphere is also true for the point magnetic dipole. (Of course, classical point dipoles are mathematical constructions rather than physical reality.)

The development for a moving Amperian electric dipole consisting of a sphere with a surface magnetic current is obviously the same as that for the Amperian magnetic dipole, and a point electric dipole formed by letting the magnetic current go to infinity as the radius of the sphere goes to zero will similarly be subject to the relativity of simultaneity. From the results of the sections on the fields of moving dipoles, it appears it doesn't matter whether the moving dipole is Amperian or Gilbertian: the fields of the moving dipoles are not the same as predicted by a naïve interpretation of the transformed magnetization-polarization tensor.

X. DISCUSSION AND CONCLUDING REMARKS

A global Lorentz transformation of the magnetic field of an at-rest, current-carrying wire to a frame of reference moving parallel to the wire results in a cylindrically symmetric radial electric field centered on the wire. This has apparently lead to the belief that the electric field experienced by a charge moving parallel to the wire is due to an actual charge density on the wire resulting from the difference in the Lorentz contraction of the spacing of charge carriers versus that of ions as seen in the moving frame ("differential Lorentz contraction").

This expectation is supported, by calculation at least, for fields associated with a wire carrying a current between a source and sink of charge [1]. However, a charge released moving parallel to a current-carrying wire and observed in its original rest frame does not move as expected, following the electric field lines in that frame. Rather, observers in that frame see an apparent contradiction between the electric field they detect and the presumed charge in the wire on the one hand and the motion of the released charge on the other. The relativity of simultaneity has fooled them into believing the wire is charged.

The relativity of simultaneity approach is able to reproduce the appearance of charge proportional to the current on a moving wire without the presence of a source or sink of charge. The result is there is no actual charge on the wire but only the appearance of charge due to relativistic timing differences between the stationary and moving reference frames. An observer located along the wire and moving with respect to it observes a different amount of charge having passed her position compared with other observers located at different points along the wire at the same time in the moving reference frame. To the moving observers there appears to be an excess of charge (positive or negative depending on the motion) on the wire due to relativistic timing differences between the moving and stationary frames.

The effect of Lorentz contraction can also be explained in terms of the relativity of simultaneity. An observer moving along the wire in the opposite direction of the current will see more charge carriers having passed her position than a trailing observer at the same time in their reference frame. Hence they will consider the charge carriers to be closer together than the ions, which are not moving as quickly in their frame. The observers must put their observations together to confirm length contraction, so the picture of the observers in the moving frame seeing themselves all equally spaced like birds on a Lorentz-contracted power line is not correct. (See the example in [19].) In the frame at rest with the wire, the moving observers arrive at the events identified in Fig. 1 at the same time, but in their frame this arrival is not simultaneous, resulting in the appearance of extra charge on the wire.

There is a problem with the idea that a charge separation on a moving Amperian magnetic dipole produces an electric dipole on the magnetic dipole due to differential Lorentz contraction and that a magnetic dipole appears on a moving electric dipole. If so, then why are the resulting fields due to the Lorentz transformation of the at-rest fields not dipolar? If there were a charge separation on a moving current loop, the electric field would look like that of an actual dipole with loops of field lines running from the positive side of the loop to the negative side, but that is not the case. Rather, the electric field is the same as that found by applying Faraday's law to the time-varying magnetic field due to the motion of the current loop.

A charge separation due to differential Lorentz contraction cannot explain the form of the electric field due to a moving current loop, and a similar problem arises for a moving electric dipole with the additional difficulty that there is no magnetic current on an electric dipole to which a differential Lorentz contraction can be applied. Rather, it seems clear that the practice of treating the dipole moments obtained by applying the Lorentz transformation to the magnetization-polarization tensor like actual dipoles in subsequent calculations is in error. This problem extends to bulk magnetic and electric materials.

Although transforming the magnetization-polarization tensor from one frame to another may be done successfully on a point-by-point basis, transforming a bulk sample into dipoles is clearly not correct. The relativity of simultaneity comes into play such that there is, for example, no actual charge separation on a bulk sample of moving magnetized material. This was illustrated by showing that the apparent charge density on a magnetized sphere in motion was due to the relativity of simultaneity. The electric field of either a current loop or a Gilbertian dipole depends only on the Lorentz transformation and not on any supposed charge distribution.

An important application of this involves the paradox of Mansuripur [6] where an observer stationary with respect to a charge in the vicinity of a current loop sees no interaction between the charge and the loop, whereas a moving observer sees the charge exert a torque on the loop. Certainly, if there were an actual charge separation on the loop, there would have to be a torque in the moving observer's reference frame. However, I have shown [23] that the apparent torque on the loop is canceled by the transformation of the angular momentum of the charge-current loop system from the stationary to the moving frame. In that work I maintained, but did not prove, there was no charge separation on the moving current loop. The missing argument for this assertion has been provided in this paper.

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