

An alternative resolution to the Mansuripur paradox

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In 2013 an article published online by the journal *Science* declared that the paradox proposed by Masud Mansuripur was resolved. This paradox concerns a point charge-Amperian magnetic dipole system as seen in a frame of reference where they are at rest and one in which they are moving. In the latter frame an electric dipole appears on the magnetic dipole. A torque is then exerted upon the electric dipole by the point charge, a torque that is not observed in the at-rest frame. Mansuripur points out this violates the relativity principle and suggests the Lorentz force responsible for the torque be replaced by the Einstein-Laub force. The resolution of the paradox reported by *Science*, based on numerous papers in the physics literature, preserves the Lorentz force but depends on the concept of hidden momentum. Here I propose a different resolution based on the overlooked fact that the charge-magnetic dipole system contains linear and angular electromagnetic field momentum. The time rate of change of the field angular momentum in the frame through which the system is moving cancels that due to the charge-electric dipole interaction. From this point of view hidden momentum is not needed in the resolution of the paradox.

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I. INTRODUCTION

In 2012 [1] Mansuripur brought attention to a paradox involving an Amperian magnetic dipole (a dipole produced by, for example, a current loop) in the vicinity a point charge. To set up the paradox, a charge and a magnetic dipole are stationary with respect to each other and arranged such that a position vector from the charge to the dipole is perpendicular to the magnetic moment of the dipole. (This is a special case. In general the magnetic moment must not lie along the line of the position vector.) Mansuripur pointed out that in the reference frame of the charge and magnetic dipole, which I will refer to as the S' frame, there would be no interaction observed between the magnetic dipole and the charge. However, he also noted that an observer in whose frame (which I'll call S) the charge-dipole arrangement moves in a direction parallel to the charge-dipole line (or with a velocity component parallel to that line) would detect a torque acting on the dipole, a torque not observed in the S' frame. This torque is due to the point charge acting on an electric dipole that is seen in the S frame by the “stationary” observer. This situation violates the principle of relativity; therefore Mansuripur suggests the Lorentz force that produces the torque should be replaced by the Einstein-Laub force [2] which does not.

This paradox was addressed earlier by Spavieri [3, 4] and is related to the paradox posed by Shockley and James [5]. Since Mansuripur's paper was published, there have been numerous papers published addressing this paradox [4, 6–10]. Many of these analyses appeal to the presence of “hidden momentum” to explain it [4, 6, 10–12]; Mansuripur, however, rejects the hidden momentum explanation in favor of Einstein-Laub [4, 8]. A news article published by the journal *Science* [12] went so far as to declare the paradox resolved, but this requires hidden momentum to exist in the charge-magnetic dipole system.

A crucial aspect of this problem is the electromagnetic linear and angular field momentum possessed by the point charge-magnetic dipole combination [13]. I intend to show that an observer in S sees the electromagnetic field angular momentum of the combination, which is constant in S' , changing in time. This changing angular momentum offsets the problematic torque identified by Mansuripur such that the total angular momentum of the combination remains constant. Trying to explain the paradox by appealing to hidden momentum, as many have done, is not correct from this point of view.

Another important aspect is the misinterpretation of the electric dipole moment seen in S being due to a Lorentz contraction effect. The conventional thinking [14] is that the electric dipole appears on the moving magnetic dipole due to differential Lorentz contraction of charged particles moving at different speeds as seen by a stationary observer. Imagine a current loop moving toward you such that the plane of the loop lies in your line of sight. That side of the loop where the positive current flows in your direction would appear to be positively charged, while the opposite side of the loop would appear to be negatively charged. In the Lorentz-contraction point of view there is an actual charge separation on the loop: in the S frame one side is *actually* positive and the other side is *actually* negative. However, this is not the way the electric dipole really arises.

It arises as a result of the relativity of simultaneity [15], not Lorentz contraction. The loop has no “*real*” charge separation, only an apparent one due to relativistic time differences between the S and S' frames at different locations on the loop. This understanding is key: If there were an actual charge separation on the loop, the resulting torque would increase the current and the magnetic moment, an unphysical effect not seen in the Lorentz transformation of the electromagnetic fields.

II. THE ELECTROMAGNETIC FIELDS AT THE POINT CHARGE IN THE S' AND S FRAMES

The paradox is set up as follows. (See figures 1 and 2.) In frame S' a loop of current of radius R lies in the $x'-y'$ plane with a positive current such that its magnetic dipole, \mathbf{m}' , is in the z' direction; that is, the positive current circulates in the positive sense about an axis parallel to the z' axis. A positive point charge q is located at the origin. The S' frame, with the charge and current loop, is moving in “standard configuration” [17], with speed v in the positive x direction of the S frame and where the space axes of the frames coincide at $t = t' = 0$. In this configuration, $y = y'$ and $z = z'$. Since the distance between the center of the loop and the point charge is a in the S frame, the center of the loop is at $(\gamma a, 0, 0)$ in the S' frame where γ is the Lorentz factor,

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}. \quad (1)$$

In frame S' the loop only produces a magnetic field. Inside and in the immediate vicinity of the loop, the magnetic field is in the positive z' direction. Everywhere else, the field is in the negative z' direction. At the position of the

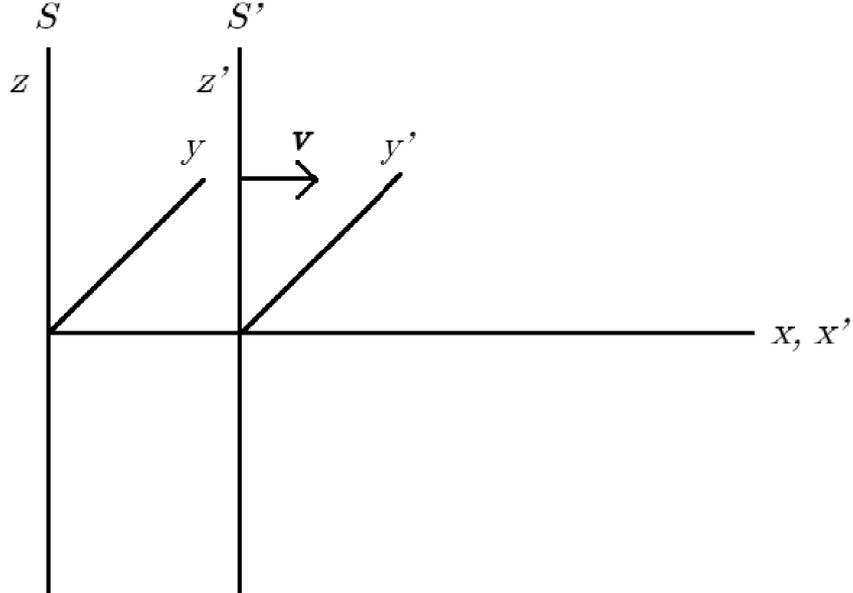


Figure 1

FIG. 1. Standard configuration of frames S and S' . S' is moving in the positive x direction with speed v where the axes of both frames coincide at $t = t' = 0$.

charge q , the field is, in SI units,

$$\mathbf{B}'_m = -\frac{\mu_o}{4\pi} \frac{m'}{\gamma^3 a^3} \hat{\mathbf{k}}. \quad (2)$$

(From now on the primes will be dropped for quantities that are the same in both reference frames. Also, some quantities that only appear with reference to the S' frame, such as the loop radius R , will be unprimed.) Lorentz-transforming the magnetic field from S' to S , you get

$$\mathbf{B}_m = \gamma \mathbf{B}'_m; \quad (3)$$

that is, the field is increased by a factor of γ and is in the same direction as in S' . The electric field at q in the S frame at $t = 0$ resulting from the transformation is

$$\mathbf{E}_m = -\frac{\mu_o}{4\pi} \frac{m'v}{\gamma^2 a^3} \hat{\mathbf{j}}. \quad (4)$$

The electric field that results from the transformation of the magnetic field from S' to S does not look like that of an actual electric dipole. For example, the field has no x component. This can be seen looking at the equation of the field,

$$\mathbf{E}_m = \frac{\gamma \mu_o m' v}{4\pi r'^3} \left[\frac{3(z^2 \hat{\mathbf{j}} - yz \hat{\mathbf{k}})}{r'^2} - \hat{\mathbf{j}} \right], \quad (5)$$

when the magnetic dipole is at the origin and where r' is the Lorentz transformed distance from the origin. Also note that since the magnetic moment is constant in the S' frame, it is also constant in the S frame. This shows that the

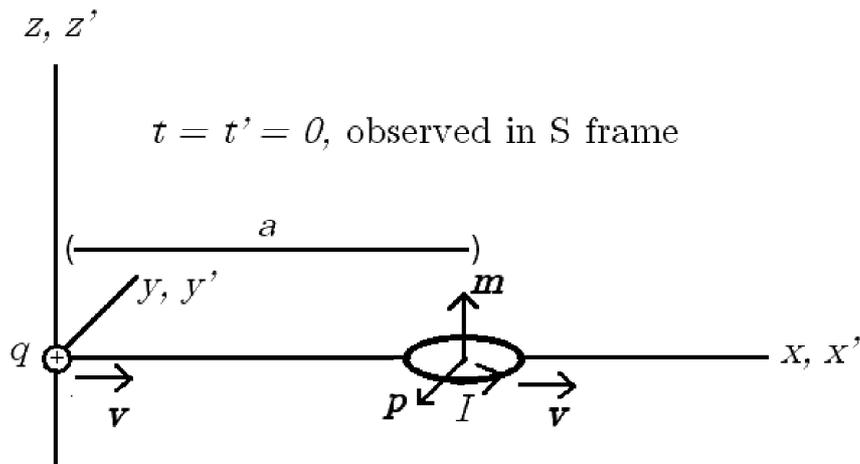


Figure 2

FIG. 2. The charge-current loop (magnetic dipole) system as seen in frame S at $t = t' = 0$. \mathbf{p} is the induced electric dipole.

magnetic moment of the current loop is not increasing, meaning the troublesome torque of Mansuripur [1] seen in the S frame is having no effect on the current loop in that frame.

With an electric field at the position of the charge q in S whereas q is stationary with respect to the magnetic dipole, it might be expected that the charge experiences a force in the S frame that it does not experience in the S' frame. Calculation of the Lorentz force on the charge shows this is not the case. Since q is moving in the positive x direction in the S frame, its current-density four-vector at $t = 0$ is

$$J^\mu = \gamma(vq\delta(\mathbf{r}), 0, 0, cq\delta(\mathbf{r})), \quad (6)$$

where $\delta(\mathbf{r})$ is the Dirac delta function and \mathbf{r} is a position vector centered at the origin of the S frame. The covariant way to calculate the force is by employing the electromagnetic field tensor $E^{\mu\nu}$ as follows, using the Einstein summation convention,

$$E^{\mu\nu} J_\nu = \begin{pmatrix} 0 & B_m & 0 & 0 \\ -B_m & 0 & 0 & -\frac{E_m}{c} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{E_m}{c} & 0 & 0 \end{pmatrix} \begin{pmatrix} -\gamma vq\delta(\mathbf{r}) \\ 0 \\ 0 \\ \gamma cq\delta(\mathbf{r}) \end{pmatrix} = \begin{pmatrix} 0 \\ \gamma vqB_m\delta(\mathbf{r}) - \gamma qE_m\delta(\mathbf{r}) \\ 0 \\ 0 \end{pmatrix}. \quad (7)$$

When Eqs. (3) and (4) are applied to this result, it is seen that the force four-vector on the charge q is zero in the S frame as it must be, since it is zero in the S' frame.

III. THE FORCE DENSITY FOUR-VECTOR ON THE CURRENT LOOP IN THE S' AND S FRAMES

First I will calculate the force four-vector on the current loop in the S' frame and then Lorentz-transform this force to the S frame. The electric field at a point $x' = \gamma a + R\cos\phi, y' = R\sin\phi$ on the loop due to the charge q , where ϕ is

the local azimuth angle measured in the positive direction from the x' axis, is given by

$$\mathbf{E}' = \frac{1}{4\pi\epsilon_o} \frac{q(\gamma\mathbf{a} + \mathbf{R})}{(\gamma^2 a^2 + R^2 + 2\gamma a R \cos\phi)^{3/2}}, \quad (8)$$

where $\mathbf{a} = a\hat{\mathbf{i}}$ and $\mathbf{R} = R(\cos\phi\hat{\mathbf{i}} + \sin\phi\hat{\mathbf{j}})$. The loop carries a current density given by

$$J^{\mu'} = \rho u(-\sin\phi, \cos\phi, 0, 0), \quad \text{that is,} \quad J_{x'} = -\rho u \sin\phi \quad \text{and} \quad J_{y'} = \rho u \cos\phi, \quad (9)$$

where ρ is the charge density of the current and u is the drift speed. (Note that I drop the four-vector notation when referring to specific components of a four-vector or four-tensor. Contravariant and covariant vectors are the same in special relativity.) Breaking up the electric field into x and y components (no z component is present at the loop) and applying the Lorentz electromagnetic field tensor, you get

$$E^{\mu'\nu'} J_{\nu'} = \begin{pmatrix} 0 & 0 & 0 & \frac{E_{x'}}{c} \\ 0 & 0 & 0 & \frac{E_{y'}}{c} \\ 0 & 0 & 0 & 0 \\ -\frac{E_{x'}}{c} & -\frac{E_{y'}}{c} & 0 & 0 \end{pmatrix} \begin{pmatrix} -J_{x'} \\ -J_{y'} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{J_{x'} E_{x'}}{c} + \frac{J_{y'} E_{y'}}{c} \end{pmatrix} \quad (10)$$

The force density in the time slot is seen to be

$$f_{ct'} = \frac{J_{x'} E_{x'}}{c} + \frac{J_{y'} E_{y'}}{c}. \quad (11)$$

The Lorentz-transformed force density four-vector in the S frame is

$$f^\mu = (\gamma \frac{v}{c} f_{ct'}, 0, 0, \gamma f_{ct'}). \quad (12)$$

Assuming the distance γa is much greater than the loop radius R , the electric field components on the loop in S' are approximately (Eq. (8))

$$E_{x'} \approx \frac{q(\gamma a + R \cos\phi)}{4\pi\epsilon_o \gamma^3 a^3} \quad \text{and} \quad E_{y'} \approx \frac{q R \sin\phi}{4\pi\epsilon_o \gamma^3 a^3}. \quad (13)$$

When you substitute $E_{x'}$ and $E_{y'}$ from the above equations and $J_{x'}$ and $J_{y'}$ from Eq. (9) into Eq. (11) and integrate over the volume, you find that the total four-force on the loop in S' is zero due to the angular dependence on ϕ . Of course, the four force must also be zero in the S frame. Nevertheless this force is responsible for the appearance of the torque that has been so troublesome, but this is not a “real” torque. Rather, in the model examined here the spatial torque originates from the Lorentz transformation of a time component in a torque four-tensor. It will turn out that this torque corresponds to an increase in the spatial angular momentum of the charge-dipole system, canceling a decrease identified by a calculation to be presented shortly. In other words, on the whole there is no torque on the system at all.

IV. THE TORQUE FOUR-TENSOR OF THE CURRENT LOOP IN S' AND S

In carrying out the calculations in this section, you assume the current loop diameter is small compared to its distance from the point charge. Then, in analogy to the formation of a point electric dipole by mathematically letting the separation between the charges go to zero as the magnitudes of the charges go to infinity, you form a point Amperian magnetic dipole from the current loop by allowing its area A to go to zero while the current I goes to infinity while holding the product $m' = IA$ constant.

Although the net four-force acting on the loop is zero in the S' frame, the components of the antisymmetric torque four-tensor, given by the volume integral

$$\tau^{\alpha\beta} = \int_V (x^\alpha f^\beta - x^\beta f^\alpha) dV, \quad (14)$$

in S' acting on the current loop do not all turn out to be zero. All components but one pair equal zero due to the ϕ dependence in the volume integrals of the torque density and the fact that $z = 0$ on the loop. The non-zero pair (symmetric-antisymmetric partners) are $\tau^{2'4'}$, the component in row 2 (the y row) and column 4 (the time column) of the tensor and $\tau^{4'2'} = -\tau^{2'4'}$. The calculation of $\tau^{2'4'}$ is carried out as follows, taking the origin about the center of the loop for the volume integration of the torque density,

$$\tau^{2'4'} = \int_{V'} (y' f_{ct'} - ct' f'_y) dV' = \int_{V'} y' f_{ct'} dV' = \int_{V'} (R \sin \phi) \left(\frac{J_{x'} E_{x'}}{c} + \frac{J_{y'} E_{y'}}{c} \right) dV'. \quad (15)$$

To perform the volume integration, you assume that the wire of the loop is one-dimensional, which lets you make the substitution $\rho dV' = \lambda R d\phi$ where λ is the linear charge density of the charge carriers responsible for the current. This allows you to write the integral as

$$\tau^{2'4'} = \frac{R^2 \lambda u}{c} \int_0^{2\pi} (-E_{x'} \sin^2 \phi + E_{y'} \sin \phi \cos \phi) d\phi. \quad (16)$$

The second integrand gives zero when integrated over ϕ . The first integrand gives

$$\tau^{2'4'} = \frac{R^2 \lambda u}{c} \int_0^{2\pi} \left(-\frac{q(\gamma a + R \cos \phi)}{4\pi \epsilon_o \gamma^3 a^3} \right) \sin^2 \phi d\phi = -\frac{\lambda(u/c) q \pi R^2}{4\pi \epsilon_o \gamma^2 a^2}. \quad (17)$$

This torque, when transformed to the S frame, gives rise to a torque about the z axis, as follows,

$$\begin{aligned} \tau_z &= \tau^{12} = \gamma \frac{v}{c} \tau^{4'2'} = \gamma \frac{v}{c} (-\tau^{2'4'}) \\ &= \frac{v}{c^2} \frac{qm}{4\pi \epsilon_o a^2}, \end{aligned} \quad (18)$$

where the magnetic moment in the S frame is $m = \gamma m'$, $m' = I \pi R^2$, and $I = \lambda u$. This is the torque, pointed out by Mansuripur [1], that is central to the paradox.

V. THE TIME RATE OF CHANGE OF THE CHARGE-DIPOLE FIELD ANGULAR MOMENTUM

According to Furry [13] the electromagnetic field angular and linear momentum associated with a point charge in the vicinity of a magnetic dipole in the S' frame (assuming the same configuration as previously) are, respectively, using the symbols in this paper,

$$\mathbf{L}' = \frac{1}{c^2} \frac{qm'}{4\pi \epsilon_o \gamma a} \hat{\mathbf{k}} \quad \text{and} \quad \mathbf{P}' = -\frac{1}{c^2} \frac{qm'}{4\pi \epsilon_o \gamma^2 a^2} \hat{\mathbf{j}}, \quad (19)$$

where the angular momentum is taken about the center of the magnet. His result for angular momentum needed no particular model for the magnet; however the result for linear momentum was derived using a spherical magnet model. It is nevertheless appropriate here for a current loop as seen by the following argument. Were the magnetic moment to decay to zero, the linear momentum stored would be entirely transmitted to the point charge [18], converting both field momenta into mechanical momenta. Hence, it is reasonable that the linear momentum should satisfy the following equation.

$$\mathbf{L}' = -\gamma a \hat{\mathbf{i}} \times \mathbf{P}'. \quad (20)$$

Comparison of this equation with \mathbf{L}' in Eq. (19) confirms that the linear momentum found by Furry is correct for this situation.

Thus the antisymmetric angular momentum tensor for the system has non-zero components $L^{1'2'}$, $L^{2'4'}$ (and their antisymmetric partners) given by

$$L_{z'} = L^{1'2'} = -L^{2'1'} = \frac{1}{c^2} \frac{qm'}{4\pi \epsilon_o \gamma a} \quad \text{and} \quad L^{2'4'} = -L^{4'2'} = -ct' P_{y'}, \quad (21)$$

where $P_{y'}$ is the linear momentum in the y' direction, given by

$$P_{y'} = -\frac{1}{c^2} \frac{qm'}{4\pi \epsilon_o \gamma^2 a^2}. \quad (22)$$

Note that the time component of the angular momentum depends on time, although the spatial angular momentum vector \mathbf{L}_z is constant in time. Transforming the tensor

$$L^{\mu'\nu'} = \begin{pmatrix} 0 & L_{z'} & 0 & 0 \\ -L_{z'} & 0 & 0 & -ct'P_{y'} \\ 0 & 0 & 0 & 0 \\ 0 & ct'P_{y'} & 0 & 0 \end{pmatrix} \quad (23)$$

to the S frame gives the component $L_z = L^{12}$ as follows,

$$L_z = \gamma L^{1'2'} + \gamma \frac{v}{c} ct' P_{y'} = \gamma(L^{1'2'} + vt' P'). \quad (24)$$

The time derivative of this gives the time rate of change of the angular momentum, which is the torque involved. Recalling that $t = \gamma t'$, this torque is

$$\frac{dL_z}{dt} = vP' = -\frac{v}{c^2} \frac{qm}{4\pi\epsilon_0 a^2}, \quad (25)$$

where the relationship $m = \gamma m'$ has been used. This torque is equal and opposite to that found in Eq. (18). Thus these torques cancel and the total angular momentum of the system is constant.

VI. DISCUSSION AND CONCLUSION

I believe some investigations into this paradox have been misled by the concept that a system containing electromagnetic momentum at rest in a certain reference frame must necessarily contain some sort of mysterious “hidden momentum” in that frame. (See, for example, [6] and [4].) However, unless you believe in some sort of magical appearance of a charge-magnetic dipole system out of the blue, there had to be some sort of assembly of this system – an interaction between the components of the system, originally with zero energy and zero momentum, and their environment to put together a system that has energy and momentum.

Here is one way the system of Mansuripur’s paradox could be put together. Assume the magnet is given by the model of Shockley and James [5] but have its rigid, charged disks not rotating initially. Now have an external agent bring a point charge from a distance into its vicinity. No net force is required to do this. The external agent then applies torque, equal and opposite to the two disks, to produce rotation, current, and a magnetic dipole, but no net mechanical angular momentum. The magnetic field will increase in the negative z direction at the location of the point charge q , producing an impulse on the charge given by $-\mathbf{P}'$ in Eq. (19) [5, 18]. To keep the charge stationary, an equal and opposite impulse must be supplied to the charge by the external agent. This impulse will store both linear and angular momentum in the electromagnetic field of the charge-dipole system, and the external agent will receive an equal and opposite amount of momentum. The total momentum is zero, but the charge-magnetic dipole does have energy and electromagnetic field momentum in its rest frame. If you like, the “hidden momentum” can be thought of as residing in the external agent.

From the result of the last section, there is no role for any hidden momentum in the charge-dipole system to account for the troublesome torque in S , as it is accounted for by the time rate of change of the transformed field angular momentum in that frame. Instead of appealing to hidden momentum, it seems preferable to interpret the paradox as follows. When you calculate the four-force acting on the current loop in its rest frame due to the presence of the point charge, there is a non-zero time component which gives rise to a time-component in the torque four-tensor. When that tensor is transformed to the frame through which the charge and loop are moving, it results in a non-zero torque, the troublesome torque of Mansuripur. However, it is also true that the charge-loop combination has linear and angular field momentum, giving rise to a time dependent component in a time slot of its angular momentum four-tensor. When the four-tensor is transformed to the frame through which the combination is moving, there results a time-dependent spatial angular momentum vector. The time derivative of this produces a torque equal and opposite to the one arising from the interaction of the point charge with the current loop. Hence the spatial angular momentum vector of the system as a whole is constant and the net torque is zero in both frames of reference.

It should also be emphasized that there is no “real” charge separation on the current loop as seen in a frame through which it is moving. An actual charge separation on the loop would result in a dipolar electric field pattern around the loop, but that is not the case. Rather, the appearance of charge separation is due to the relativity of simultaneity [15] not differential Lorentz contraction. Hence, there is no rotational acceleration associated with the torque and no change in the loop’s magnetic moment. This would not be true were one side of the loop actually positive and the

other negative. Were this the case it would be difficult to understand how the torque due to the point charge was negated by the electromagnetic field torque.

Finally, I should point out that nothing in this work says anything about the physical validity of the Lorentz versus the Einstein-Laub formalisms for a classical Amperian magnetic dipole. Mansuripur has pointed out that when an electromagnetic system is treated as a whole both formalisms give the same total force and torque [19]. Mansuripur's argument is that the Lorentz formalism needs to be corrected by including the angular momentum density of the magnetization of the magnetic dipole that results from Einstein-Laub, $\mathbf{r} \times (\epsilon_0 \mathbf{E} \times \mathbf{M})$, where \mathbf{M} is the magnetization. This term cancels the supposed hidden angular momentum density to resolve the paradox [19]. As it turns out, this term "fixes" the problem by accounting for the lack of consideration of the role of the field angular momentum when the Lorentz formalism is not properly applied. Hence, there is no paradox in either the Lorentz or Einstein-Laub formalisms if the magnetic dipole is Amperian. Hidden momentum is not needed in either.

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